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STOCHASTIC ORDERING AND A CLASS OF MULTIVARIATE NEW BETTER THAN--ETC(U)

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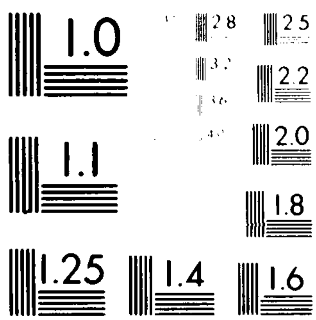

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STOCHASTIC ORDERING AND A CLASS OF MULTIVARIATE NEW BETTER THAN USED DISTRIBUTIONS

by

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A new characterization for the univariate class of new better than used (NBU) distributions in terms of stochastic ordering is introduced. A multivariage version of this characterization is then used to define a multivariate class of NBU distributions. Basic properties of this class are derived. Comparisons and relationships of this new class with earlier classes are developed. Two multivariate new worse than used (NWU) classes of life distributions are defined and compared and their basic properties are studied.

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ABSTRACT

A new characterization for the univariate class of new better than used (NBU) distributions in terms of stochastic ordering is introduced. A multivariate version of this characterization is then used to define a multivariate class of NBU distributions. Basic properties of this class are derived. Comparisons and relationships of this new class with earlier classes are developed. Two multivariate new worse than used (NWU) classes of life distributions are defined and compared and their basic properties are studied.

1. INTRODUCTION AND SUMMARY

The univariate "new better than used" (NBU) class of life distributions was shown by Marshall and Proschan (1972) to play a key role in the study of maintenance policies. Basic properties of this class and its relationships to other univariate classes have been developed (see Barlow and Proschan (1975)).

In this paper a new characterization for the

univariate NBU class is introduced in terms of stochastic ordering between appropriate random variables. A natural multivariate extension of this characterization is used to define a class of multivariate NBU distributions (or equivalently random vectors). The advantage of this characterization is to permit us to utilize the well known properties of stochastic ordering to derive the various preservation properties of this new class of multivariate NBU distributions. In particular, simpler proofs are given for some well known closure properties of the univariate NBU class. Moreover such approach adds more probabilistic flavor to the treatment of the NBU class both in the univariate and multivariate cases.

Other versions of the multivariate NBU distributions have been introduced and studied. See, e.g. El-Neweihi, Proschan and Sethuraman (1980), and Marshall and Shaked (1979). The multivariate NBU class given in this paper is shown to be a proper subclass of the class given by Marshall and Shaked except of course in the univariate case where the two classes coincide.

In section 2 the multivariate NBU class of life distributions is defined and its properties are studied. In section 3, we compare our multivariate NBU class

with other classes of multivariate NBU distributions. We also introduce the concept of NBU n-dimensional processes and derive a simple closure property. Finally in section 4, two multivariate classes of "new worse than used" (NWU) life distributions are introduced and some of their properties are studied.

Throughout, the term "increasing" is used to mean "non-decreasing" and "decreasing" is used to mean "non-increasing". A function f defined on R^n is said to be increasing if it is increasing in each argument. Vectors in R^n are denoted by $\underline{x} = (x_1,\ldots,x_n)$ and $\underline{x} \le \underline{y}$ means $x_i \le y_i$, $i = 1,\ldots,n$. Given two vectors $\underline{x}$ and $\underline{y}$ in R^n $\underline{x} \wedge \underline{y}$ denotes $(x_1 \wedge y_1,\ldots,x_n \wedge y_n)$ where $x_i \wedge y_i$ denotes $\min(x_i,y_i)$, $i = 1,\ldots,n$. The vector $\underline{x} \vee \underline{y}$ is similarly defined. Given a vector $\underline{x} \in R^n$, $Q_{\underline{x}} = \{\underline{y}: \underline{x} < \underline{y}\}$, where $\underline{x} < \underline{y}$ means $x_i < y_i$, $i = 1,\ldots,n$. The set $\{\underline{x}: \underline{0} \le \underline{x}\}$ is denoted by R^n_+, where $\underline{0} = (0,\ldots,0)$.

A random variable $\underline{X} = (X_1,\ldots,X_n)$ is said to be stochastically smaller than the random variable Y (denoted by $X \overset{st}{\le} Y$) if $P(X > x) \le P(Y > x)$ for every real number x. The random vector $\underline{X} = (X_1,\ldots,X_n)$ is said to be stochastically smaller than $\underline{Y} = (Y_1,\ldots,Y_n)$ (denoted by $\underline{X} \overset{st}{\le} \underline{Y}$) if $f(\underline{X}) \overset{st}{\le} f(\underline{Y})$ for every real-valued, borel measurable and increasing function f defined on R^n.

If $\underline{X}$ and $\underline{Y}$ have the same distribution then we write $\underline{X} \overset{st}{=} \underline{Y}$. It is well known that $\underline{X} \overset{st}{\leq} \underline{Y}$ if and only if $P(\underline{X} \in U) \leq P(\underline{Y} \in U)$ for every upper open subset U of R^n. (A set $U \subseteq R^n$ is said to be an upper set if $\underline{x} \in U$ and $\underline{x} \leq \underline{y}$ implies that $\underline{y} \in U$. (A lower set is similarly defined).

We need only consider sets U of the form $\bigcup_{i=1}^{m} Q_{\underline{x}^i}$, where $\underline{x}^1,\ldots,\underline{x}^m$ are m vectors in R^n. Such sets are called fundamental upper domains (see Block and Savits (1979).

The following are well known preservation properties for stochastic ordering that will be used throughout this paper:

(i) Let $\underline{X} \overset{st}{\leq} \underline{Y}$ and $f: R^n \longrightarrow R^m$ be an increasing borel measurable function. Then $f(\underline{X}) \overset{st}{\leq} f(\underline{Y})$.

(ii) Let $\{\underline{X}^1,\ldots,\underline{X}^m\}$ and $\{\underline{Y}^1,\ldots,\underline{Y}^m\}$ be two sets of independent random vectors, where $\underline{X}^i,\underline{Y}^i$ are n_i-dimensional vectors, $i = 1,\ldots,m$. Suppose $X^i \overset{st}{\leq} \underline{Y}^i$, $i = 1,\ldots m$. Then $(\underline{X}^1,\ldots,\underline{X}^m) \overset{st}{\leq} (\underline{Y}^1,\ldots,\underline{Y}^m)$.

(iii) Suppose $\underline{X}^i \overset{st}{\leq} Y^i$, $i = 1,2,\ldots$. If $\underline{X}^i$ converges to $\underline{X}$ in distribution (denoted by $\underline{X}^i \xrightarrow{d} \underline{X}$) and $\underline{Y}^i \xrightarrow{d} \underline{Y}$, then $\underline{X} \overset{st}{\leq} \underline{Y}$.

Throughout the remainder of this paper all the

random variables (random vectors) considered are assumed to be nonnegative.

2. A MULTIVARIATE NEW BETTER THAN USED CLASS OF LIFE DISTRIBUTIONS AND ITS PROPERTIES

In this section a multivariate new better than used class of life distributions (MNBU) is introduced and its properties are studied.

First recall that a random variable X or its corresponding distribution function F is said to be <u>NBU</u> if

$$\bar{F}(s + t) \leq \bar{F}(s)\bar{F}(t) \quad \text{for all} \quad s,t \geq 0, \tag{2.1}$$

where $\bar{F} = 1 - F$.

The following Lemma gives a characterization of the NBU property.

<u>Lemma 2.1.</u> A random variable X is NBU if and only if

$$X \overset{st}{\leq} \left(\frac{X'}{\alpha} \wedge \frac{X''}{1-\alpha}\right) \quad \text{for every} \quad 0 < \alpha < 1, \tag{2.2}$$

where X' and X'' are independent and $X \overset{st}{=} X' \overset{st}{=} X''$.

<u>Proof</u>. It is easy to verify that (2.1) is equivalent to

$F(x) \leq F(\alpha x)F(1-\alpha)x)$ for every real number x and every $0 < \alpha < 1$. The relation (2.2) follows readily by observing that $P\left(\left(\frac{X'}{\alpha} \wedge \frac{X''}{1-\alpha}\right) > x\right) = F(\alpha x)F\left(1-\alpha\right)x\right)$. ||

The natural multivariate extension of (2.2) is now used to define the MNBU class.

<u>Definition 2.2</u>. A random vector $\underline{X} = (X_1,\ldots,X_n)$ is said to be a <u>MNBU</u> random vector if $\underline{X} \overset{st}{\leq} \frac{\underline{X}'}{\alpha} \wedge \frac{\underline{X}''}{1-\alpha}$ for every $0 < \alpha < 1$, where $\underline{X}'$ and $\underline{X}''$ are independent and $\underline{X} \overset{st}{=} \underline{X}' \overset{st}{=} \underline{X}''$.

The class of all such MNBU random vectors (of all orders) is called the MNBU class. The following theorem shows that the MNBU class enjoys many desirable properties. Utilizing the well known properties of stochastic ordering enables us to derive the properties of the MNBU class with considerable ease. In particular, simple proofs are given for the closure of this class under convolution and formation of coherent systems.

<u>Theorem 2.3</u>. The following properties hold for the MNBU class:

(P1) Let T be an NBU random variable. Then T is 1-dimensional MNBU.

(P2) Let $\underline{T} = (T_1,\dots,T_n)$ be MNBU. Then $(T_{i_1},\dots,T_{i_k})$ is k-dimensional MNBU, $1 \le i_1 < \dots < i_k \le n$, $k = 1,\dots,n$.

(P3) Let $\underline{T} = (T_1,\dots,T_n)$ be MNBU. Then $(T_{\pi_1},\dots,T_{\pi_n})$ is MNBU where $\pi_1,\dots,\pi_n$ is a permutation of $1,\dots,n$.

(P4) Let $\underline{T} = (T_1,\dots,T_n)$ be MNBU and $a_i > 0$, $i = 1,\dots,n$. Then $\sum_{i=1}^{n} a_i T_i$ is NBU.

(P5) Let $\underline{T} = (T_1,\dots,T_n)$ be MNBU and $a_i > 0$, $i = 1,\dots,n$. Then $\min_{1 \le i \le n} a_i T_i$ is NBU.

(P6) Let $\underline{T}$ be MNBU and $\tau_1,\dots,\tau_m$ be m life functions of coherent systems of order n each. Then $(\tau_1(\underline{T}),\dots,\tau_m(\underline{T}))$ is m-dimensional MNBU.

(P7) Let $\underline{T} = (T_1,\dots,T_n)$ be MNBU. Let $f\colon R^n \longrightarrow R^m$ be an increasing nonnegative borel measurable function such that $f\left(\frac{\underline{x}}{\alpha}\right) \le \frac{f(\underline{x})}{\alpha}$ for all $\underline{x} \in R^n$ and all $0 < \alpha < 1$. Then $f(\underline{T})$ is MNBU.

(P8) Let $\underline{T}^1,\dots,\underline{T}^k$ be independent MNBU vectors of dimension $n_1,\dots,n_k$ respectively. Then $(\underline{T}^1,\dots,\underline{T}^k)$ is $(n_1+\dots+n_k)$-dimensional MNBU.

(P9) Let $\underline{T}^1,\dots,\underline{T}^k$ be independent MNBU vectors of order n each. Then $\underline{T}^1 + \dots + \underline{T}^k$ is n-dimensional MNBU.

(P10) Let $\underline{T}^i$, $i = 0,1,\ldots$ be a sequence of MNBU random vectors of the same order n. Assume $\underline{T}^i \xrightarrow{d} \underline{T}^0$. Then $\underline{T}^0$ is MNBU .

Proof. (P1) is obvious.

(P2) through (P7). Since (P2) through (P6) are special cases of P(7) we need only prove (P7).

Let $\underline{T}'$ and $\underline{T}''$ be two independent copies of $\underline{T}$ and $0 < \alpha < 1$. By the given properties of f and the well known properties of stochastic ordering we can easily verify that $f(\underline{T}) \overset{st}{\leq} \dfrac{f(\underline{T}')}{\alpha} \wedge \dfrac{f(\underline{T}'')}{1-\alpha}$. The random vectors $f(\underline{T}')$ and $f(\underline{T}'')$ are two independent copies of $f(\underline{T})$ and consequently $f(\underline{T})$ is MNBU.

(P8) Let $(\underline{T}^1)', (\underline{T}^1)'', (\underline{T}^2)', (\underline{T}^2)'', \ldots, (\underline{T}^k)', (\underline{T}^k)''$ be mutally independent random vectors such that $\underline{T}^i \overset{st}{=} (\underline{T}^i)' \overset{st}{=} (\underline{T}^i)''$, $i = 1,\ldots,k$. Let $0 < \alpha < 1$. Then $\underline{T}^i \overset{st}{\leq} \dfrac{(\underline{T}^i)'}{\alpha} \wedge \dfrac{(\underline{T}^i)''}{1-\alpha}$, $i = 1,\ldots,k$. Since $\{\underline{T}^1,\ldots,\underline{T}^k\}$ and $\left\{\dfrac{(\underline{T}^1)'}{\alpha} \wedge \dfrac{(\underline{T}^1)''}{1-\alpha}, \ldots, \dfrac{(\underline{T}^k)'}{\alpha} \wedge \dfrac{(\underline{T}^k)''}{1-\alpha}\right\}$ are two sets of independent random vectors we have

$$(\underline{T}^1;\ldots;\underline{T}^k) \overset{st}{\leq} \left(\frac{(\underline{T}^1)'}{\alpha} \wedge \frac{(\underline{T}^1)''}{1-\alpha}, \ldots, \frac{(\underline{T}^k)'}{\alpha} \wedge \frac{(\underline{T}^k)''}{1-\alpha}\right) =$$

$$\frac{((\underline{T}^1)';\ldots;(\underline{T}^k)')}{\alpha} \wedge \frac{(\underline{T}^1)'';\ldots;(\underline{T}^k)''}{1-\alpha} .$$

The result now follows by observing that $((T^1)',\ldots,(T^k)')$ and $((\underline{T}^1)''\ldots,(T^k)'')$ are two independent copies of $(\underline{T}^1,\ldots,\underline{T}^k)$.

(P9) The proof follows readily by (P7) and (P8).

(P10) Let $(\underline{T}^i)'$ and $(T^i)''$ be two independent copies of $\underline{T}^i$, $i = 0,1,2,\ldots$ and let $0 < \alpha < 1$. Since $\underline{T}^i \overset{st}{\leq} \frac{(\underline{T}^i)'}{\alpha} \wedge \frac{(\underline{T}^i)''}{1-\alpha}$, $i = 1,2,\ldots$ and $\underline{T}^i \xrightarrow{d} \underline{T}^0$ and $\frac{(\underline{T}^i)'}{\alpha} \wedge \frac{(\underline{T}^i)''}{1-\alpha} \xrightarrow{d} \frac{(\underline{T}^0)'}{\alpha} \wedge \frac{(\underline{T}^0)''}{1-\alpha}$ we have $\underline{T} \overset{st}{\leq} \frac{(\underline{T}^0)'}{\alpha} \wedge \frac{(\underline{T}^0)''}{1-\alpha}$ and consequently $\underline{T}^0$ is MNBU. ||

Remark 2.4. The closure of the univariate NBU class under the formation of coherent systems and convolution follow now as special cases of (P6),(P8) and (P9).

Remark 2.5. In view of theorem 2.3 one can construct and identify various examples of MNBU random vectors. In particular let $T_1,\ldots,T_n$ be independent NBU random variables and let $\phi \neq S_i \subset \{1,\ldots,n\}$, $i = 1,\ldots,m$, then

(i) $(T_1,\ldots,T_n)$ is MNBU .

(ii) If $T_i^* = \min_{j \in S_i} T_j$, $i = 1,\ldots,m$, then $(T_1^*,\ldots,T_m^*)$ is MNBU. When the T_i's are exponential the random vector $(T_1^*,\ldots,T_m^*)$ is the well known multivariate

exponential (MVE) of Marshall and Olkin (1967).

(iii) If $T_i^* = \sum_{j \in S_i} T_j$, $i = 1,\ldots,m$, then $(T_1^*,\ldots,T_m^*)$ is MNBU.

(iv) The random vector $(T_{(1)},\ldots,T_{(n)})$ of order statistics corresponding to $T_1,\ldots,T_n$ is MNBU.

3. OTHER CLASSES OF MULTIVARIATE NEW BETTER THAN USED DISTRIBUTIONS AND THEIR RELATION TO THE MNBU CLASS.

Various multivariate extensions of the univariate NBU class are now available (see El-Neweihi, Proschan and Sethuraman (1980) and Marshall and Shaked (1979)). Each of these classes satisfies some of the desired properties which one would reasonably expect for a class of multivariate NBU distributions. In this section we compare the MNBU with most of these other classes. We also introduce and discuss briefly a concept of NBU vector-valued stochastic processes. A more detailed discussion of this concept and its relation to the multivariate NBU classes will be given in another paper (El-Neweihi, 1980).

Consider nonnegative random variables $T_1,\ldots,T_n$ whose minimum is not degenerate at 0 and whose joint

distribution satisfies one of the following conditions:

(A) $T_1, \ldots, T_n$ are independent and each T_i is an NBU random variable.

(B) The random vector $\underline{T} = (T_1, \ldots, T_n)$ has a representation $T_i = \min_{j \in S_i} X_j$, where the random variables $X_1, \ldots, X_m$ are independent NBU random variables and $\emptyset \neq S_i \subset \{1, \ldots, m\}$, $i = 1, \ldots, n$ and $\bigcup_{i=1}^{n} S_i = \{1, \ldots, m\}$.

(C) $\underline{T}$ is MNBU.

(D) For every open upper set $A \subset R_+^n$ and for every $\alpha > 0$, $\beta > 0$ we have

$$P(\underline{T} \in (\alpha+\beta)A) \leq P(\underline{T} \in \alpha A) P(\underline{T} \in \beta A). \tag{3.1}$$

(E) For all $a_i > 0$, $i = 1, \ldots, n$, $\min_{1 \leq i \leq n} a_i T_i$ is NBU.

(F) For each $\emptyset \neq A \subset \{1, \ldots, n\}$, $\min_{i \in A} T_i$ is an NBU random variable.

(G) Each T_i is an NBU random variable.

Each of the classes of multivariate distributions defined by (A)-(G) may be designated as a class of multivariate NBU distributions. Our MNBU class is quite analogous to the class defined by (D) and these

two classes are the only ones among the other classes that enjoy all the properties in theorem 2.3. We now compare these classes. In a recent paper by El-Neweihi, Proschan and Sethuraman (1980) it was shown that $A \Rightarrow (B) \Rightarrow (E) \Rightarrow (F) \Rightarrow (G)$ and no other implication among these five classes is possible. We now show that $(B) \Rightarrow (C) \Rightarrow (D) \Rightarrow (E)$ and no other implication among these four classes is possible. In view of theorem 2.3 $(B) \Rightarrow (C)$ is obvious. Also since the class defined by (D) satisfy all the properties in theorem 2.3, $(D) \Rightarrow (E)$ is obvious. In the following lemma we show that $(C) \Rightarrow (D)$

<u>Lemma 3.1</u>. Let $\underline{T}$ be MNBU. Then $\underline{T}$ satisfies condition (D).

<u>Proof</u>. Let A be an upper open subset of R_+^n and $\alpha > 0$ and $\beta > 0$. Let $\alpha' = \frac{\alpha}{\alpha+\beta}$ and let $\underline{T}'$ and $\underline{T}''$ be two independent copies of $\underline{T}$. Since $(\alpha+\beta)A$ is an upper set we have

$P(\underline{T} \in (\alpha+\beta)A) \leq P(\frac{\underline{T}}{\alpha'} \wedge \frac{\underline{T}''}{1-\alpha} \in (\alpha+\beta)A)$. Also

$$P(\frac{\underline{T}'}{\alpha'} \wedge \frac{\underline{T}''}{1-\alpha'} \in (\alpha+\beta)A) \leq P(\frac{\underline{T}'}{\alpha'} \in (\alpha+\beta)A)P(\frac{\underline{T}''}{1-\alpha'} \in (\alpha+\beta)A)$$

$$= P(\underline{T} \in \alpha A)P(\underline{T} \in \beta A),$$

since $\underline{T}'$ and $\underline{T}''$ are independent copies if $\underline{T}$. ||

We now give examples to show that no other implication among the classes (B),(C),(D) and (E) is possible. First we need the following lemma

Lemma 3.2. Let $\underline{T} = (T_1,\ldots,T_n)$ be a random vector and let $\overline{F}(\underline{t}) = P(T_1 > t_1,\ldots,T_n > t_n)$ for all $\underline{t} \in R^n$. Then $\underline{T}$ is MNBU if and only if

$$\sum_{i=1}^{m} \overline{F}(\underline{t}^i) - \sum_{1\leq i\leq j\leq m} \overline{F}(\underline{t}^i \vee \underline{t}^j) + \ldots + (-1)^{m-1}\overline{F}(\underline{t}^1 \vee \ldots \vee \underline{t}^m)$$

$$\leq \sum_{i=1}^{m} \overline{F}(\alpha\underline{t}^i)\overline{F}((1-\alpha)\underline{t}^i) - \sum_{1\leq i\leq j\leq m} \overline{F}(\alpha(\underline{t}^i \vee t^j))\overline{F}(1-\alpha)(\underline{t}^i \vee \underline{t}^j)) +$$

$$+ (-1)^{m-1}\overline{F}(\alpha(\underline{t}^1 \vee \ldots \vee \underline{t}^m))\overline{F}((1-\alpha)(\underline{t}^1 \vee \ldots \vee \underline{t}^m)), \quad (3.2)$$

for every m and every $0 < \alpha < 1$ and every $\underline{t}^1,\ldots,\underline{t}^m \in R^n$.

Proof. By known properties of stochastic ordering $\underline{T}$ is MNBU if and only if $P(\underline{T} \in \bigcup_{i=1}^{m} Q_{\underline{t}^i}) \leq P(\frac{\underline{T}'}{\alpha} \wedge \frac{\underline{T}''}{1-\alpha} \in \bigcup_{i=1}^{m} Q_{\underline{t}^i})$ for every $0 < \alpha < 1$, and all m vectors $\underline{t}^1,\ldots,\underline{t}^m$ in R^n and every m. The result in (3.2) follows now immediately by the inclusion-exclusion principle. ||

Example 3.3. Let $\bar{F}(x,y) = e^{-\sqrt{x^2+y^2}}$, $x \geq 0$ and $y \geq 0$. It is easy to show that $\bar{F}(x,y)$ satisfies (3.2). Indeed if $\underline{T} = (T_1,T_2)$ is the bivariate random vector whose joint survival function is $\bar{F}(x,y)$, then $\underline{T} \stackrel{st}{=} \frac{\underline{T}'}{\alpha} \wedge \frac{\underline{T}''}{1-\alpha}$ for every $0 < \alpha < 1$, where $\underline{T}'$ and $\underline{T}''$ are two independent copies of $\underline{T}$. It was shown in the paper by El-Neweihi, et al (1980) that $\underline{T}$ cannot satisfy B. Thus $C \not\Rightarrow B$.

Example 3.4. Consider $(T_1,T_2) = (U,1-U)$ where U is uniformly distributed on the unit interval. It was shown by Block and Savits (1978) that (T_1,T_2) is MIFRA. Since the MIFRA class of distributions is contained in the class defined by (D) we have (T_1,T_2) must satisfy (D). Now to show that (T_1,T_2) does not satisfy (C) take $\underline{t}^1 = (0,\frac{1}{4})$ and $\underline{t}^2 = (\frac{1}{4},0)$. The left hand side of (3.2) is 1 and the right hand side is $1 - \frac{\alpha(1-\alpha)}{8}$. Thus $(D) \not\Rightarrow (C)$.

Example 3.5. Suppose (T_1,T_2) has density

$$f(t_1,t_2) = \frac{32}{47} \quad \text{if } t_1 \geq 0,\ t_2 \geq 0 \text{ and}$$
$$t_1 + t_2 \leq \frac{1}{4}$$

$$= \frac{64}{47} \text{ if } 0 \le t_1 \le 1,\ 0 \le t_2 \le 1 \text{ and }$$

$$t_1 + t_2 \ge \frac{3}{4}$$

$$= 0 \quad \text{elsewhere .}$$

According to Esary and Marshall (1979) $\min(a_1T_1, a_2T_2)$ is NBU for every $a_1 > 0$, $a_2 > 0$. We now show that $\max(T_1, T_2)$ is not NBU. To see this note that $\bar{F}(\frac{1}{4}) = \bar{F}(\frac{3}{8}) = \frac{1}{47}$ and $\bar{F}(\frac{1}{8}) < 1$ consequently $\bar{F}(\frac{3}{8}) > \bar{F}(\frac{1}{4})\bar{F}(\frac{1}{8})$. Now since the class defined by (D) is closed under formation of coherent systems (T_1, T_2) does not satisfy (D). Thus (E) $\not\Rightarrow$ (D) .

We conclude this section by giving a multivariate extension of the concept of NBU process due to El-Neweihi, Proschan and Sethuraman (1978). The generalized NBU closure theorem given by these authors is also extended.

Let $\{\underline{X}(t),\ t \ge 0\}$ be nonnegative decreasing right-continuous n-dimensional stochastic process. For every lower closed set $C \subset R^n$ let $T^C = \inf\{t: \underline{X}(t) \in C\}$. (With the convention $\inf \phi = +\infty$).

<u>Definition 3.6</u>. The process $\{\underline{X}(t),\ t \ge 0\}$ is said to be <u>NBU</u> if T^C is NBU for all C.

When $n = 1$ the above definition coincides with the one given by El-Neweihi, et al (1978).

The following theorem gives a preservation property of vector-valued NBU processes.

Theorem 3.7. Let $\{\underline{X}(t), t > 0\}$ be NBU process and let ϕ be a real-valued increasing nonnegative left-continuous function on R_+^n. Then $\{\phi(\underline{X}(t)), t \geq 0\}$ is a 1-dimensional NBU process.

Proof. First observe that if U is an open upper subset of R^n then $\{\underline{X}(t) \in U\} = \{T^{U^c} > t\}$ for all $t \geq 0$. Now let $T^a = \inf\{t:\phi(X(t)) \leq a\}$ and note that $\{\phi(\underline{X}(t)) > a\} = \{T^a > t\}$ for all $a \in R^1$ and all $t \geq 0$. Let $0 < \alpha < 1$ be given then we have
$P\{T^a > t\} = P\{\phi(\underline{X}(t)) > a\} = P\{\underline{X}(t) \in U\} = P\{T^{U^c} > t\} \leq$
$P\{T^{U^c} > \alpha t\}P\{T^{U^c} > (1-\alpha)t\} = P\{\underline{X}(\alpha t) \in U\}\ P\{\underline{X}((1-\alpha)t) \in U\} =$
$P\{T^a > \alpha t\}P\{T^a > (1-\alpha)t\}$, where $U = \{\underline{y}:\phi(\underline{y}) > a\}$ is an upper open subset of R^n. Thus T^a is NBU for every a and $\{\phi(\underline{X}(t), t \geq 0\}$ is a 1-dimensional NBU process.

In the context of multistate coherent systems $\{\underline{X}(t), t \geq 0\}$ represents the components performance process and ϕ represent a montone structure function. Thus theorem 3.7. can be viewed as a generalized NBU closure theorem (under formation of monotone systems).

4. MULTIVARIATE NEW WORSE THAN USED CLASSES OF LIFE DISTRIBUTIONS.

The univariate "new worse than used" (NWU) class of life distributions arises naturally in the study of shock models and maintenacne policies (see Barlow and Proschan (1975)). Among the most interesting properties of this class is its closure under formation of certain mixtures.

Each of the multivariate classes of NBU life distributions discussed in section 3 can be associated naturally with a corresponding class of multivariate NWU life distributions. In this section only two such classes are defined and their properties are studied. One of these classes is shown to be a proper subclass of the other. A multivariate version of the closure of one of these classes under special mixtures is given.

We begin by giving the following two definitions.

Definition 4.1. The random vector $\underline{T} = (T_1,\ldots,T_n)$ is said to be a MNWU if $\frac{\underline{T}'}{\alpha} \wedge \frac{\underline{T}''}{1-\alpha} \overset{st}{\leq} \underline{T}$ for all $0 < \alpha < 1$, where $\underline{T}'$ and $\underline{T}''$ are two independent copies of $\underline{T}$.

Definition 4.2. The random vector $\underline{T} = (T_1,\ldots,T_n)$ is said to be a multivariate NWU if

$P(\underline{T} \in A) \geq P(\underline{T} \in \alpha A)P(\underline{T} \in (1-\alpha)A)$, for every $0 < \alpha < 1$ and every open upper set $A \subset R^n_+$.

The class of all MNWU vectors is called the MNWU class and the class of all vectors satisfying definition 4.2. is called the strong MNWU class and is denoted by SMNWU.

Lemma 4.3. Let $\underline{T}$ be a SMNWU vector then $\underline{T}$ is a MNWU vector.

Proof. Let $0 < \alpha < 1$ and A be an upper open set. Then

$$P\left(\frac{\underline{T}'}{\alpha} \wedge \frac{\underline{T}''}{1-\alpha} \in A\right) \leq P\left(\frac{\underline{T}'}{\alpha} \in A \text{ and } \frac{\underline{T}''}{1-\alpha}\right) =$$

$$P(\underline{T} \in \alpha A)P(\underline{T} \in (1-\alpha)A) \leq P(\underline{T} \in A).$$

Thus $\underline{T}$ is a MNWU vector. ||

The following example shows that there exists a MNWU vector which is not MNWU.

Example 4.4. Let (T_1,T_2) be the bivariate random vector whose survival pure distribution function is given by $\bar{F}(x,y) = e^{-\sqrt{x^2+y^2}}$, $x \geq 0$, $y \geq 0$. It was shown in example 3.3. that $\underline{T} \overset{st}{=} \frac{\underline{T}}{\alpha} \wedge \frac{\underline{T}''}{1-\alpha}$ for every $0 < \alpha < 1$ and consequently $\underline{T}$ is MNWU. Now to see that $\underline{T}$ is

not SMNWU. Let $A = Q_{(1,0)} \cup Q_{(0,1)}$ and let $0 < \alpha < 1$. Now $P(\underline{T} \in A) = 2e^{-1} - e^{-\sqrt{2}}$ and $P(\underline{T} \in \alpha A)P(\underline{T} \in (1-\alpha)A) = 4e^{-1} + e^{-\sqrt{2}} - 2e^{-(\alpha + (1-\alpha)\sqrt{2})}$ $- 2e^{-(1-\alpha)+\alpha\sqrt{2})}$. The fact that e^{-x} is a strictly convex function shows that $P(\underline{T} \in A) < P(\underline{T} \in \alpha A)P(\underline{T} \in (1-\alpha)A)$. The random vector $\underline{T}$ is therefore not a SMNWU vector.

The following theorem gives properties enjoyed by the MNWU class.

<u>Theorem 4.5</u>. The following properties hold for the MNWU class:

(P1) Let T be an NWU random variable. Then T is 1-dimensional MNWU.

(P2) Let $\underline{T} = (T_1,\ldots,T_n)$ be MNWU. Then $(T_{i_1},\ldots,T_{i_k})$ is k-dimensional MNWU, $1 \leq i_1 \leq i_2 < \ldots < i_k \leq n$, $k = 1,\ldots,n$.

(P3) Let $\underline{T} = (T_1,\ldots,T_n)$ be MNWU. Then $(T_{\pi_1},\ldots,T_{\pi_n})$ is MNWU, where $\pi_1,\ldots,\pi_n$ is a permutation of $\{1,\ldots,n\}$.

(P4) Let $\underline{T} = (T_1,\ldots,T_n)$ be MNWU and $a_i > 0$, $i = 1,\ldots,n$. Then $\min_{1\leq i\leq n} a_iT_i$ is NWU.

(P5) Let $\underline{T} = (T_1,\ldots,T_n)$ be MNWU. Let $g: R^n \longrightarrow R^m$ be an increasing nonnegative borel measurable function such that $\frac{g(\underline{x})}{\alpha} \wedge \frac{g(\underline{y})}{1-\alpha} \leq g(\frac{x}{\alpha} \wedge \frac{y}{1-\alpha})$ for all $x, y \in R^n$ and all $0 < \alpha < 1$. Then $g(\underline{T})$ is MNWU.

(P6) Let $\underline{T}^1,\ldots,\underline{T}^k$ be independent MNWU vectors of dimension $n_1,\ldots,n_k$ respectively. Then $(\underline{T}^1,\ldots,\underline{T}^k)$ is $(n_1+n_2+\ldots+n_k)$ - dimensional MNWU.

(P7) Let $\underline{T}^i$, $i = 0,1,..$ be a sequence of MNWU vectors of the same order n. Assume $\underline{T}^i \xrightarrow{d} \underline{T}^0$. Then $\underline{T}^0$ is MNWU.

Proof. The proof is similar to that of theorem 2.3. and is therefore omitted.

Remark 4.6. Note that properties (P2)-(P4) are all special important cases of (P5).

Remark 4.7. In view of (P5) of theorem 2.3 and (P4) of theorem 4.5, a random vector $\underline{T}$ which is both MNBU and MNWU has the property that $\min_{1\leq i\leq n} a_i T_i$ is exponential for all $a_i > 0$, $i = 1,\ldots,n$. The random vector in example 4.4 and the MVE of Marshall and Olkin (1967) are examples of such random vectors.

Before we study the properties of the SMNWU class we first observe that a random vector $\underline{T}$ is SMNWU if

and only if

$$E\,\phi(\underline{T}) \geq E\,\phi\left(\frac{\underline{T}}{\alpha}\right) E\,\phi\left(\frac{\underline{T}}{1-\alpha}\right), \qquad (4.1)$$

where ϕ is a binary increasing borel measurable function on R^n and $0 < \alpha < 1$ (EX denotes as usual the expected value of the random variable X).

Theorem 4.8. The following properties hold for the SMNWU class:

(P1) Let T be an NWU random variable. Then T is 1-dimensional SMNWU.

(P2) Let $\underline{T} = (T_1,\ldots,T_n)$ be a SMNWU vector. Let $g: R^n \longrightarrow R^m$ be nonnegative increasing borel measurable function such that $\frac{g(\underline{x})}{\alpha} \leq g\left(\frac{\underline{x}}{\alpha}\right)$ for all $\underline{x} \in R^n$ and every $0 < \alpha < 1$.

(P3) Let $\underline{T}^i$, $i = 0,1,\ldots$ be a sequence of SMNWU vectors of the same order n. Assume $\underline{T}^n \xrightarrow{d} \underline{T}^0$. Then $\underline{T}^0$ is SMNWU .

Proof. (P1) is obvious.

(P2) Let ϕ be a binary increasing borel measurable function on R^m and $0 < \alpha < 1$. Then

$E\,\phi(\frac{g(\underline{T})}{\alpha})\,E\,\phi(\frac{g(\underline{T})}{1-\alpha}) \le E\,\phi(g(\frac{\underline{T}}{\alpha}))\,E\,\phi(g(\frac{\underline{T}}{1-\alpha})) \le$ $E\,\phi(g(\underline{T}))$, where the first equality follows from properties of g and the second inequality follows from the property of $\underline{T}$.

(P3) It suffices to show that $P(\underline{T}^0 \in \bigcup_{i=1}^{\ell} Q_{\underline{t}}i) \ge P(\underline{T}^0 \in \alpha(\bigcup_{i=1}^{\ell} Q_{\underline{t}}i))P(\underline{T}^0 \in (1-\alpha)(\bigcup_{i=1}^{\ell} Q_{\underline{t}}i))$, for every ℓ vectors $\underline{t}^1,\dots,\underline{t}^{\ell}$, $\ell = 1,2,\dots$ and every $0 < \alpha < 1$. Let $0 < \alpha < 1$ be arbitrary but fixed number. Let $\mathcal{C} = \{A: \bigcup_{i=1}^{\ell} Q_{\underline{t}}i,\ \underline{t}^1,\dots,\underline{t}^{\ell} \in R^n,\ \ell = 1,2,\dots$ and $P(\underline{T}^0 \in \partial A) = P(\underline{T}^0 \in \alpha(\partial A)) = P(\underline{T}^0 \in (1-\alpha)(\partial A)) = 0\}$ where ∂A denotes the boundary of the set A. It is not hard to show that given $B = \bigcup_{i=1}^{r} Q_{\underline{t}}i$, there exists a sequence $A_m \in \mathcal{C}$ such that $A_1 \subset A_2 \subset \dots$ and $B = \bigcup_{m=1}^{\infty} A_m$. Now since $\underline{T}^n$ is SMNWU we have $P(\underline{T}^n \in A_m) \ge P(\underline{T}^n \in \alpha A_m)P(\underline{T}^n \in (1-\alpha)A_m), n = 1,2,\dots$. Taking limits as $n \longrightarrow \infty$ then as $m \longrightarrow \infty$ we get $P(\underline{T}^0 \in B) \ge P(\underline{T}^0 \in \alpha B)P(\underline{T}^0 \in (1-\alpha)B)$. Since α and B are arbitrarily chosen we have $\underline{T}^0$ is SMNWU. ||

Remark 4.9. Obvious choices of particular functions g in (P2) of theorem 4.8. show that the SMNWU class enjoys properties (P2)-(P4) of theorem 4.5.

Remark 4.10. Let $\underline{T} = (T_1, T_2)$ be a bivariate random vector where T_1 and T_2 are independent exponential random variables with parameter 1 each. It can be shown that $\underline{T}$ is not SMNWU. This shows that the SMNWU does not have the important property (P6) of theorem 4.5.

We conclude this section by demonstrating that a certain subclass of SMNWU class is preserved under mixtures. Recall that if $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ is a family of distribution functions on R^n and G is a probability measure defined on a σ-Algebra of subsets of Λ, then $F(\underline{t}) = \int F_\lambda(\underline{t})\, dG(\lambda)$ is a distribution function on R^n. (It is assumed that for every fixed $\underline{t}$, $F_\lambda(\underline{t})$ is a measurable function on Λ). The distribution F is called a mixture of distributions from $\mathcal{F}$.

The following theorem is well known for the case $n = 1$ (see Barlow and Proschan (1975)).

Theorem 4.11. Suppose F is a mixture of F_λ, $\lambda \in \Lambda$, with each F_λ NWU, and no distinct $F_\lambda, F_{\lambda'}$, crossing on $(0,\infty)$. Then F is NWU.

The following theorem gives a multivariate extension of theorem 4.11.

Theorem 4.12. Suppose F is a mixture of F_λ, $\lambda \in \Lambda$. Let μ, μ_λ, $\lambda \in \Lambda$ be the corresponding probability measures on $\mathcal{B}^n$ the borel σ-field of R^n. Suppose that for each $\lambda, \lambda' \in \Lambda$ either $\mu_\lambda(A) \leq \mu_{\lambda'}(A)$ for all upper borel sets A of R^n or the reverse inequality holds. Suppose that for each λ, μ_λ is the probability measure induced by a SMNWU random vector. Then μ is the probability measure induced by a SMNWU random vector.

Proof. First note that if A is an open upper subset of R_+^n, then $\mu_\lambda(A)$ is a measurable function of λ and $\mu(A) = \int_\Lambda \mu_\lambda(A)dG(\lambda)$, where G is a probability measure on Λ. Let $0 < \alpha < 1$ be given and A be an upper open subset of R_+^n. Then $\mu_\lambda(\alpha A)$, $\mu_\lambda(1-\alpha)A)$ are two similarly ordered functions of λ, and by Chebyshev inequality for similarly ordered functions (Hardy, Littlewood and Po'lya, (1952)) we have

$$\int_\Lambda \mu_\lambda(\alpha A)\mu_\lambda((1-\alpha)A)dG(\lambda) \geq \left(\int_\Lambda \mu_\lambda((\alpha A)dG(\lambda)\right)\left(\int_\Lambda \mu_\lambda((1-\alpha)A)dG(\lambda)\right).$$

It follows that $\mu(A) \geq \mu(\alpha A)\mu((1-\alpha)A)$. ||

BIBLIOGRAPHY

Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

Block, H.W. and Savits, T.H. (1978). The Class of MIFRA Lifetimes and Its Relation to Other Classes. Unpublished report.

Block, H.W. and Savits, T.H. (1979). Multivariate IFRA distributions. Ann. Probability, to appear.

El-Neweihi, E., Proschan, F. and Sethuraman, J. (1978). Multistate Coherent Systems, J. Appl. Prob., 15, 675-688.

El-Neweihi, E., Proschan, F. and Sethuraman, J. (1980). A Multivariate New Better Than Used Class Derived from a Shock Model. Unpublished report.

El-Neweihi, E. (1980).Vector-Valued NBU Processes. In preparation

Esary, J.D. and Marshall, A.W. (1979). Multivariate Distributions with Increasing Hazard Rate Average. Ann. Probability, 7, 359-370.

Hardy, G.H., Littlewood, J.E. and Po lya, G. (1952). Inequalities. London: Cambridge University Press.

Marshall, A.W. and Olkin, I. (1967). A Multivariate Exponential Distribution. J. Amer. Statist. Assoc., 62,30-44.

Marshall, A.W. and Proschan, F. (1972). Classes of Distributions Applicable in Replacement, with Renewal Theory Implications. Proc. 6th Berkeley Symp. Math. Stat. Prob., ed. by L. LeCam, J. Neyman and E.L. Scott, 395-415. Univ. of California Press, Berkeley, California.

Marshall, A.W. and Shaked, M. (1979). A Class of Multivariate New Better Than Used Distributions. Unpublished report.

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